ELSEVIER
Comput. Methods Appl. Mech. Engrg. 178 (1999) 257–262
Computer methods in applied mechanics and engineering
www.elsevier.com/locate/cma

Homotopy perturbation technique

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Received 10 May 1998

Abstract

The homotopy perturbation technique does not depend upon a small parameter in the equation. By the homotopy technique in topology, a homotopy is constructed with an imbedding parameter $p \in [0, 1]$, which is considered as a "small parameter". Some examples are given. The approximations obtained by the proposed method are uniformly valid not only for small parameters, but also for very large parameters. © 1999 Elsevier Science S.A. All rights reserved.

Keywords: Perturbation techniques; Homotopy; Nonlinearity

1. Introduction

In the last two decades with the rapid development of nonlinear science, there has appeared ever-increasing interest of scientists and engineers in the analytical techniques for nonlinear problems. The widely applied techniques are perturbation methods. But, like other nonlinear analytical techniques, perturbation methods have their own limitations. At first, almost all perturbation methods are based on an assumption that a small parameter must exist in the equation. This so-called small parameter assumption greatly restricts applications of perturbation techniques. As is well known, an overwhelming majority of nonlinear problems have no small parameters at all. Secondly, the determination of small parameters seems to be a special art requiring special techniques. An appropriate choice of small parameters leads to ideal results. However, an unsuitable choice of small parameters results in bad effects, sometimes seriously. Furthermore, the approximate solutions solved by the perturbation methods are valid, in most cases, only for the small values of the parameters. It is obvious that all these limitations come from the small parameter assumption.

In this paper, the author will first propose a new perturbation technique coupled with the homotopy technique. The proposed method, requiring no small parameters in the equations, can readily eliminate the limitations of the traditional perturbation techniques.

2. Basic idea of homotopy perturbation method

To illustrate the basic ideas of the new method, we consider the following nonlinear differential equation

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PII: S0045-7825(99)00018-3

$$A(u) - f(\boldsymbol{r}) = 0, \quad \boldsymbol{r} \in \Omega \tag{1}$$

with boundary conditions

$$B(u, \partial u/\partial n) = 0, \quad \boldsymbol{r} \in \Gamma, \tag{2}$$

where A is a general differential operator, B is a boundary operator, $f(\boldsymbol{r})$ is a known analytic function, Γ is the boundary of the domain Ω.

The operator A can, generally speaking, be divided into two parts L and N, where L is linear, while N is nonlinear, Eq. (1), therefore, can be rewritten as follows

$$L(u) + N(u) - f(\boldsymbol{r}) = 0. \tag{3}$$

By the homotopy technique [1,2], we construct a homotopy $v(r, p) : \Omega \times [0, 1] \to \mathbb{R}$ which satisfies

$$\mathscr{H}(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(\boldsymbol{r})] = 0, \quad p \in [0, 1], \ \boldsymbol{r} \in \Omega \tag{4a}$$

or

$$\mathscr{H}(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(\boldsymbol{r})] = 0, \tag{4b}$$

where $p \in [0, 1]$ is an embedding parameter, u_0 is an initial approximation of Eq. (1), which satisfies the boundary conditions. Obviously, from Eq. (4) we have

$$\mathscr{H}(v, 0) = L(v) - L(u_0) = 0, \tag{5}$$

$$\mathscr{H}(v, 1) = A(v) - f(\boldsymbol{r}) = 0, \tag{6}$$

the changing process of p from zero to unity is just that of $v(\boldsymbol{r}, p)$ from $u_0(\boldsymbol{r})$ to $u(\boldsymbol{r})$. In topology, this is called deformation, and $L(v) - L(u_0)$, $A(v) - f(\boldsymbol{r})$ are called homotopic.

In this paper, the present author will first use the imbedding parameter p as a "small parameter", and assume that the solution of Eq. (4) can be written as a power series in p:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \tag{7}$$

Setting $p = 1$ results in the approximate solution of Eq. (1):

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \tag{8}$$

The coupling of the perturbation method and the homotopy method is called the homotopy perturbation method, which has eliminated limitations of the traditional perturbation methods. In the other hand, the proposed technique can take full advantage of the traditional perturbation techniques.

The series (8) is convergent for most cases, however, the convergent rate depends upon the nonlinear operator $A(v)$ (the following opinions are suggested by an unknown referee):

1. The second derivative of $N(v)$ with respect to v must be small, because the parameter p may be relatively large, i.e. $p \to 1$.
2. The norm of $L^{-1}\partial N/\partial v$ must be smaller than one, in order that the series converges.

3. Some simple examples

3.1. Example 1

At first, we will consider the Lighthill equation [3], for it is widely studied by the PLK method (Poincaré–Lightill–Kuo Method). The equation can be written as follows:

$$(x + \varepsilon y)\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0, \quad y(1) = 1. \tag{9}$$

We can readily construct a homotopy which satisfies

$$(1-p)\left[\varepsilon Y\frac{\mathrm{d}Y}{\mathrm{d}x}-\varepsilon y_0\frac{\mathrm{d}y_0}{\mathrm{d}x}\right]+p\left[(x+\varepsilon Y)\frac{\mathrm{d}Y}{\mathrm{d}x}+Y\right]=0,\quad p\in[0,1]. \tag{10}$$

One may now try to obtain a solution of Eq. (10) in the form

$$Y(x)=Y_0(x)+pY_1(x)+p^2Y_2(x)+\cdots, \tag{11}$$

where the $Y_i(x),\ i=0,1,2,\ldots$ are functions yet to be determined. The substitution of Eq. (11) into Eq. (10) yields

$$\varepsilon Y_0\frac{\mathrm{d}Y_0}{\mathrm{d}x}-\varepsilon y_0\frac{\mathrm{d}y_0}{\mathrm{d}x}=0, \tag{12}$$

$$\varepsilon Y_1\frac{\mathrm{d}Y_1}{\mathrm{d}x}+\left[(x+\varepsilon Y_0)\frac{\mathrm{d}Y_0}{\mathrm{d}x}+Y_0\right]=0. \tag{13}$$

The initial approximation $Y_0(x)$ or $y_0(x)$ can be freely chosen, here we set

$$Y_0(x)=y_0(x)=-x/\varepsilon,\qquad Y_0(1)=-1/\varepsilon \tag{14}$$

so that the residual of Eq. (9) at $x=0$ vanishes. The substitution of Eq. (14) into Eq. (13) yields

$$\varepsilon Y_1\frac{\mathrm{d}Y_1}{\mathrm{d}x}-\frac{x}{\varepsilon}=0,\quad Y_1(1)=1+1/\varepsilon. \tag{15}$$

The solution of Eq. (15) may be written as follows

$$Y_1=\frac{1}{\varepsilon}\sqrt{x^2+2\varepsilon+\varepsilon^2}. \tag{16}$$

If the first approximation is sufficient, then we obtain

$$y_1(x)=Y_0(x)+Y_1(x)=\frac{1}{\varepsilon}(-x+\sqrt{x^2+2\varepsilon+\varepsilon^2}) \tag{17}$$

which is the exact solution.

3.2. Example 2

In this example, a more complicated equation [4] is considered where the PLK method will not be valid, which reads

$$(x^n+\varepsilon y)\frac{\mathrm{d}y}{\mathrm{d}x}+nx^{n-1}y=mx^{m-1},\quad y(1)=b>1, \tag{18}$$

where $n=2,3,4,\ldots$, $m=0,1,2,3,\ldots$

A homotopy can be readily constructed as follows

$$(1-p)\left[\varepsilon Y\frac{\mathrm{d}Y}{\mathrm{d}x}-\varepsilon y_0\frac{\mathrm{d}y_0}{\mathrm{d}x}\right]+p\left[(x^n+\varepsilon Y)\frac{\mathrm{d}Y}{\mathrm{d}x}+nx^{n-1}Y-mx^{m-1}\right]=0,\quad p\in[0,1]. \tag{19}$$

By the same manipulation as Example 1, we set $Y_0(x)=y_0(x)=-x^n/\varepsilon$ so that the residual of Eq. (18) at $x=0$ is zero. We, therefore, have

$$\varepsilon Y_1\frac{\mathrm{d}Y_1}{\mathrm{d}x}+\left[(x^n+\varepsilon Y_0)\frac{\mathrm{d}Y_0}{\mathrm{d}x}+nx^{n-1}Y_0-mx^{m-1}\right]=0,\quad Y_1(1)=b+\frac{1}{\varepsilon} \tag{20a}$$

or

$$\varepsilon Y_1\frac{\mathrm{d}Y_1}{\mathrm{d}x}+\left[-nx^{2n-1}/\varepsilon-mx^{m-1}\right]=0,\quad Y_1(1)=b+\frac{1}{\varepsilon}. \tag{20b}$$

The solution of Eq. (20b) can be easily obtained

$$Y_1(x) = \sqrt{\frac{1}{\varepsilon^2}x^{2n} + \frac{2}{\varepsilon}x^m + b^2 + \frac{2b-2}{\varepsilon}}. \tag{21}$$

The first-order approximation of Eq. (18), therefore, can be obtained as follows

$$y_1(x) = Y_0(x) + Y_1(x) = -\frac{x^n}{\varepsilon} + \sqrt{\frac{1}{\varepsilon^2}x^{2n} + \frac{2}{\varepsilon}x^m + b^2 + \frac{2b-2}{\varepsilon}} \tag{22}$$

which is also the exact solution!

It should be pointed out that the approximate solution of Eq. (18) obtained by the PLK method is not uniformly valid. For details, please see the discussion in Ref. [5].

3.3. Example 3

In this example, the well-known Duffing equation [3] will be studied which can be expressed as follows

$$\frac{d^2u}{dt^2} + u + \varepsilon u^3 = 0, \quad u(0) = A, \ u'(0) = 0. \tag{23}$$

We construct a homotopy which satisfies

$$L(v) - L(u_0) + pL(u_0) + p\varepsilon v^3 = 0, \tag{24}$$

where $Lu = d^2u/dt^2 + u$.

By the same manipulation as before, we have the following linear systems

$$L(v_0) - L(u_0) = 0, \quad v_0(0) = A, \ v_0'(0) = 0, \tag{25}$$

$$L(v_1) + L(u_0) + \varepsilon v_0^3 = 0, \quad v_1'(0) = v_1(0) = 0. \tag{26}$$

We set $v_0(t) = u_0(t) = A\cos\alpha t$ with an unknown constant α as the initial approximation of Eq. (24). Therefore from Eq. (26), we have

$$\frac{d^2v_1}{dt^2} + v_1 + A\left(-\alpha^2 + 1 + \frac{3}{4}\varepsilon A^2\right)\cos\alpha t + \frac{\varepsilon A^3}{4}\cos 3\alpha t = 0. \tag{27}$$

The solution of Eq. (27) can be readily obtained by the so-called variational iteration method [6,7]

$$\begin{aligned} v_1(t) &= \int_0^t \sin(\tau - t)\left[A(-\alpha^2 + 1 + \frac{3}{4}\varepsilon A^2)\cos\alpha\tau + \frac{\varepsilon A^3}{4}\cos 3\alpha\tau\right]d\tau \\ &= \left(-\alpha^2 + 1 + \frac{3}{4}\varepsilon A^2\right)\frac{A}{\alpha^2 - 1}(\cos\alpha t - \cos t) + \frac{\varepsilon A^3}{4(9\alpha^2 - 1)}(\cos 3\alpha t - \cos t). \end{aligned} \tag{28}$$

The constant α can be identified by various methods such as method of weighted residuals (least square method, method of collocation, Galerkin method). In this paper, we will use a very simple technique to determine the constant. In order to eliminate the secular term which may occur in the next iteration, we set the coefficient of $\cos t$ zero:

$$-\left(-\alpha^2 + 1 + \frac{3}{4}\varepsilon A^2\right)\frac{A}{\alpha^2 - 1} - \frac{\varepsilon A^3}{4(9\alpha^2 - 1)} = 0 \tag{29a}$$

or

$$\alpha = \sqrt{\frac{10 + 7\varepsilon A^2 + \sqrt{64 + 104\varepsilon A^2 + 49\varepsilon^2 A^4}}{18}}. \tag{29b}$$

For small ε, Eq. (29b) can be approximately expressed as $\alpha = \sqrt{1 + 3\varepsilon A^2/4 + O(\varepsilon^2)}$. Eq. (28), therefore, can be re-written as follows

$$v_1(t) = \left(-\alpha^2 + 1 + \frac{3}{4}\varepsilon A^2 \right) \frac{A}{\alpha^2 - 1} \cos \alpha t + \frac{\varepsilon A^3}{4(9\alpha^2 - 1)} \cos 3\alpha t \tag{30}$$

with α defined as Eq. (29b).

If, for example, the first-order approximation is sufficient, then we have

$$u_1(t) = v_0(t) + v_1(t) = \frac{3\varepsilon A^3}{4(\alpha^2 - 1)} \cos \alpha t + \frac{\varepsilon A^3}{4(9\alpha^2 - 1)} \cos 3\alpha t. \tag{31}$$

The period of the solution can be expressed as follows

$$T = \frac{2\pi}{\alpha} \quad \text{with } \alpha \text{ defined as (29b)} \tag{32}$$

while the exact period reads Ref. [3]

$$T_{\text{ex}} = \frac{4}{\sqrt{1+\varepsilon}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k\,\sin^2 x}} \quad \text{with } k = \frac{\varepsilon A^2}{2(1 + \varepsilon A^2)} \tag{33}$$

and the period obtained by the perturbation method is Ref. [3]

$$T_{\text{pert}} = \frac{2\pi}{1 + 3\varepsilon A^2/8}. \tag{34}$$

It is interesting to notice that Eq. (34) is valid only for small ε, whereas Eqs. (31) and (32) for a very large region $0 \leqslant \varepsilon < \infty$. Furthermore the approximations obtained by the proposed new method are of high accuracy, even when $\varepsilon \to \infty$, we have

$$\lim_{\varepsilon \to \infty} \frac{T_{\text{ex}}}{T} = \frac{2\sqrt{7/9}}{\pi} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - 0.5\,\sin^2 x}} = \frac{2\sqrt{7/9}}{\pi} \times 1.68575 = 0.946.$$

Therefore, for any value of ε, it can be easily proved that the maximal relative error is less than 5.4%.

4. Conclusion

In this paper we have studied few problems with or without small parameters with the homotopy perturbation technique. The results show that:

1. The proposed method does not require small parameters in the equations, so the limitations of the traditional perturbation methods can be eliminated.
2. The initial approximation can be freely selected with possible unknown constants.
3. The approximations obtained by this method are valid not only for small parameters, but also for very large parameters. Furthermore their first-order approximations are of extreme accuracy. Although few examples given in this paper are nonlinear differential equations, it can be applicable to nonlinear partial differential equations.

Acknowledgements

The work is supported by National Science Foundation of China and Shanghai Education Foundation for Young Scientists. The author wishes to thank an unknown referee for his valuable discussions, and part of his suggestion has been directly cited.

References

[1] S.J. Liao, An approximate solution technique not depending on small parameters: a special example, Int. J. Non-Linear Mechanics 30 (3) (1995) 371–380.

[2] S.J. Liao, Boundary element method for general nonlinear differential operators, Engineering Analysis with Boundary Element 20 (2) (1997) 91–99.

[3] A.H. Nayfeh, Introduction to Perturbation Techniques, Wiley, New York, 1981.

[4] C.C. Lin, Mathematics Applied to Deterministic Problems in Natural Sciences, Macmillan, New York, 1974.

[5] Y.B. Wang et al., An Introduction to Perturbation Techniques (in Chinese), Shanghai Jiaotong University Press, 1986.

[6] J.H. He, A new approach to nonlinear partial differential equations, Communications in Nonlinear Science and Numerical Simulation 2 (4) (1997) 230–235.

[7] J.H. He, Nonlinear oscillation with fractional derivative and its approximation, International Conference on Vibration Engineering '98, 1998, Dalian, China.