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A CALCULATION OF THE VISCOUS FORCE EXERTED BY A FLOWING FLUID ON A DENSE SWARM OF PARTICLES

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Summary

A calculation is given of the viscous force, exerted by a flowing fluid on a dense swarm of particles. The model underlying these calculations is that of a spherical particle embedded in a porous mass. The flow through this porous mass is decribed by a modification of Darcy's equation. Such a modification was necessary in order to obtain consistent boundary conditions. A relation between permeability and particle size and density is obtained. Our results are compared with an experimental relation due to Carman.

§ 1. *Introduction.* The viscous force on a spherical particle moving through a fluid is given by Stokes' well known formula

$$K = 6\pi\eta R v_0 \tag{1}$$

where

K is the force exerted by the fluid on the particle,

η is the fluid viscosity,

R is the radius of the spherical particle,

v_0 is the velocity of flow at a large distance from the particle.

In this paper a modification of (1) will be derived, valid for a dense swarm of particles. The same problem was treated by J. M. Burgers[1]) in a wholly different way.

§ 2. *A modification of Darcy's equation.* The mean fluid flow through the swarm of particles is calculated by considering the swarm as a porous mass.

An empirical relation describing the flow of a fluid through a porous mass is D a r c y's equation [2])

$$\mathbf{v} = -\frac{k}{\eta}\,\mathrm{grad}\, p \tag{2}$$

where

v is the rate of flow through a surface element of unit area,
k is the permeability of the porous mass,
p is the pressure.

It should be noted that v pertains to the mean rate of flow.

Now the viscous force on a particle will be calculated by considering the fluid flow around this particle in detail, while the influence of the other particles is represented by a porous mass in which the chosen particle is embedded. Equation (2), however, cannot be used as such. A first objection is that no viscous stress tensor has been defined in relation to it. The viscous shearing stresses acting on a volume element of fluid have been neglected; only the damping force of the porous mass ($\eta v/k$) has been retained. This is a good approximation for small permeabilities. In our problem, however, an equation has to be found which retains its validity for low particle densities ($k \to \infty$). Related to this objection are the difficulties encountered in framing boundary conditions for problems of fluid flow through porous masses and adjoining empty space.

Let the fluid flow in the porous mass be described by (2), combined with the condition of incompressibility

$$\mathrm{div}\,\mathbf{v} = 0, \tag{3}$$

and in empty space by the well known Navier Stokes equation, neglecting inertial terms,

$$\mathrm{grad}\, p = \eta \Delta \mathbf{v} \tag{4}$$

combined with (3). Now (4) is a second order differential equation while (2) is of the first order. This makes it impossible to formulate rational boundary conditions.

The following way out of these difficulties is suggested. An equation is set up, stating the equilibrium between the forces acting on a volumeelement of fluid, i.e. the pressure gradient, the divergence of the viscous stress tensor and the damping force caused by the porous mass.

It is suggested to modify (2) in the following way:

$$\operatorname{grad} p = -\frac{\eta}{k}\mathbf{v} + \eta' \Delta \mathbf{v}. \tag{5}$$

This equation has the advantage of approximating (2) for low values of k and (4) for high values of k.

It should be noted that $\mathbf{v}$ in this equation pertains to the mean velocity in the porous mass in the same way as is the case for (2). The factor η' in the term $\eta'\Delta\mathbf{v}$ (i.e. the divergence of the stresstensor) may be different from η. Its value will be discussed later on. The permeability k usually has such a small value that the last term in (5) is relatively unimportant. This, however, is not the case in our problem.

The boundary conditions between a porous mass and a hole may now be derived from (5). Consider a small volume-element partly in the hole. Let its dimension normal to the wall of the hole tend to zero. Then the force resulting from the damping term $(\eta v/k)$ is negligible in comparison to the normal and shearing stresses. Therefore the following components of the stress tensor should be continuous:

$$p_{nt} = \eta'\left(\frac{\partial v_n}{\partial t} + \frac{\partial v_t}{\partial n}\right), \tag{6}$$

$$p_{nn} = -p + 2\eta'\frac{\partial v_n}{\partial n}, \tag{7}$$

where n indicates the normal direction and t the tangential direction.

From (3) it follows that the normal component of the velocity should be continuous. A fourth boundary condition may be derived for the tangential velocity-component v_t. The discountinuous boundary between porous mass and hole may be replaced by a very small transition region with a permeability varying from k in the porous mass to ∞ in the hole.

Assuming (5) to remain valid in the transition region, it follows that $\partial v_t/\partial n$ is finite and continuous. By making the transition region infinitesimally small it follows that v_t is continuous at the boundary. It should be emphasized that this conclusion is only valid if the various assumptions about the boundary are justified.

§ 3. *Calculation of a modified Stokes' formula for a swarm of particles.* Solutions of (3) and (5) have now to be found, subject to the following conditions. Infinitely far removed from the spherical particle the fluid flow is a parallel flow in the x direction with a velocity v_0. At the boundary of the particle conditions may be derived by imagining an infinitesimally small slit between the particle and the porous mass. It follows that v_n and v_t should be zero, while the viscous forces acting on the particle may be derived from (6) and (7).

Now a solution of (3) and (5) satisfying these conditions is

$$v = \operatorname{grad}\left[\left(v_0 r + a\,\frac{e^{-\lambda r}(1+\lambda r) - 1}{\lambda^2 r^2} - \frac{b}{r^2}\right)\cos\vartheta\right] + a\,\frac{e^{-\lambda r}}{r}\,\mathbf{i} \tag{8}$$

where r, ϑ and φ are polar coordinates with their axis in the x-direction,

$$\lambda = \left\{\frac{\eta}{k\eta'}\right\}^{\frac{1}{2}},$$

$$a = -\tfrac{3}{2}\, v_0 R\, e^{\lambda R},$$

$$b = \tfrac{1}{2}\, v_0 R\left\{-R^2 + \frac{3}{\lambda^2}\left(e^{\lambda R} - 1 - \lambda R\right)\right\}.$$

Here $\mathbf{i}$ is the unit-vector in the x-direction and R is the radius of the particle.

Neglecting a constant term we find for the pressure:

$$p = \eta'\left[-\lambda^2 v_0 r + \frac{a}{r^2} + \frac{\lambda^2 b}{r^2}\right]\cos\vartheta. \tag{9}$$

Now the stress components at the particle are found from

$$p_{rr} = -p + 2\eta'\,\frac{\partial v_r}{\partial r}, \tag{10}$$

$$p_{r\vartheta} = \eta'\left(r\,\frac{\partial v_\vartheta/r}{\partial r} + \frac{1}{r}\,\frac{\partial v_r}{\partial \vartheta}\right). \tag{11}$$

The force K on the particle is derived by integrating the x-components of (10) and (11) over the sphere:

$$K = \frac{4\pi}{3} R^2\,(\bar{p}_{rr} - 2\bar{p}_{r\vartheta}), \tag{12}$$

where $\bar{p}_{rr}$ and $\bar{p}_{r\vartheta}$ are the radial parts of p_{rr} and $p_{r\vartheta}$.

Substitution of (10), (11) and (8) yields

$$K = 6\pi\eta' v_0 R\,[1 + \lambda R + \lambda^2 R^2/3]. \tag{13}$$

This is the analogue of Stokes' formula (1) with correction terms. For infinite permeability (i.e. low particle density, $k \to \infty$, $\lambda \to 0$) Stokes' formula is obtained as a limiting form.

In order to make (13) applicable a relation has to be derived between λ (which contains the permeability k) and the particle size and density. This relation is obtained in the following way.

Consider a column of length l and transverse section of area O, containing N particles distributed at random. Now a fluid flowing through this column obeys Darcy's law

$$\Delta p/l = \eta v_0/k \tag{14}$$

where all quantities are taken positive. The total force exerted by the fluid on the particles (when the force on the walls of the column may be neglected) is

$$\Delta p \,.\, O = \eta v_0 V/k \tag{15}$$

where $V = lO$ is the total volume of the column.

But according to (13):

$$\Delta p \,.\, O = 6\pi\eta' v_0 RN \left[1 + \lambda R + \frac{\lambda^2 R^2}{3}\right]. \tag{16}$$

Combination of (15) and (16) gives an equation for λR. Its solution is

$$\lambda R = \left[9 + 3\left(8\frac{V}{V_0} - 3\right)^{\frac{1}{2}}\right] \Big/ \left(4\frac{V}{V_0} - 6\right) \tag{17}$$

where $V_0 = (4\pi/3)\,R^3 N$ is the total volume of the particles. This is a relation between the permeability and V_0/V. The quantity $1 - V_0/V$ is called the porosity.

§ 4. *Comparison with experiment.* The total force exerted on the particles is found by substitution of (17) in (16). This leads to the relation

$$\frac{6\pi\eta R v_0 N}{\Delta p \,.\, O} = \frac{\eta'}{\eta}\left[1 + \frac{3V_0}{4V}\left(1 - \sqrt{\frac{8V}{V_0} - 3}\right)\right\}. \tag{18}$$

The dimensionless quantity in the left hand member of (18) must become equal to 1 for small values of V_0/V. This means that Stokes' law (1) should be valid when the particles are far apart. For a denser swarm of particles (18) becomes much smaller than (1).

The precise nature of the formula still depends upon the assumption made concerning η'. We might attempt to substitute the Einstein formula as an approximation:

$$\eta' = \eta\left(1 + 2.5\,\frac{V_0}{V}\right), \tag{19}$$

valid for the viscosity of fluid containing a suspension of particles [2]).

On the other hand a comparison with experiment may be obtained by comparing (18) to an experimental relation formulated by Carman [3]). This relation was set up for a column packed with particles. In such conditions the particles will not contribute to the transport of momentum in the fluid, they may even hinder this transport. Therefore (19) is not applicable; η' may even be smaller than η. We chose $\eta' = \eta$.

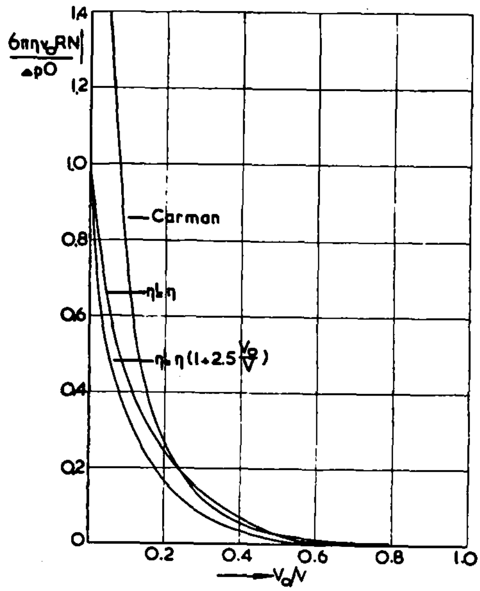

Fig. 1.

Translated into our symbols the Carman relation reads:

$$\frac{6\pi\eta v_0 RN}{\Delta p \,.\, O} = \frac{V(1 - V_0/V)^3}{10\,V_0}\,. \tag{20}$$

This relation was based on experiments for values of V_0/V in the region of 0.5 — 0.6.

For small values of V_0/V it certainly is not correct: it should tend to 1 as our relation does, while in fact it tends to infinity.

In fig. 1 our relation (18) for a swarm of particles (η' given by (19)) and packed particles ($\eta' = \eta$) and Carman's relation (20) are indicated.

A comparison for various values of V_0/V is obtained by dividing the right hand member of (20) by that of (18) as shown in the following table.

TABLE I

V_0/V	(20)/(18)
0	∞
0.1	1.75
0.2	1.08
0.3	0.88
0.4	0.86
0.5	1.10
0.6	3.12
0.7	3.73

For $V_0/V < 0.6$ there is satisfactory agreement between our formula and the experimental evidence.

For high values of V_0/V (> 0.6) our relation gives considerably lower values than Carman's. In the neighbourhood of $V_0/V = 0.7$ our relation even yields a minimum. This is caused by the fact that relation (17) gives zero permeability for $V_0/V = 2/3$. In this region our model of one particle embedded in a porous mass is too schematic. A better approximation might be obtained by considering the fluctuations in permeability caused by the arrangement of the neighbouring particles round the chosen one. This, however, would involve very complicated calculations.

Burgers' formula (loc. cit. formula (64))

$$\frac{6\pi\eta v_0 RN}{\Delta p\, O} = \frac{1}{1 + \left\{-1 + 4.9\left(\frac{3V_0}{4\pi V}\right)^{1/3} + 0.67\left(\frac{4\pi V}{3V_0}\right)^{2/3}\right\}\frac{V_0}{V}}$$

gives a numerical result which is about 20 times as large as ours for $V_0/V = 0.5$. We have the impression that this is too large.

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Note added in proof. A discussion with Prof. J. M. Burgers on equation (5) led to the following conclusions.

Equation (5) should be regarded as a more or less arbitrary interpolation in the region where the damping force and the viscous force are of the same magnitude.

The introduction of a damping force is justified by the fact that the particles are supported by exterior forces. These forces are gravity and mechanical forces transmitted by contacts.

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